Isoparametric hypersurfaces in S^{13} with six principal curvatures

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Abstract

This paper provides a final remark on the classification of isoparametric hypersurfaces in S^{13} with six principal curvatures. Although a paper has appeared claiming to present a "counterexample", we show that this claim arises from a misunderstanding and does not contradict the author's argument. Consequently, the original classification remains correct and fully valid as stated. [1]

1 Introduction

The Appendix of the referenced text [3] claims the existence of counterexample to Proposition 6.2 in the author's arcicle [2]. In the present paper, the author clarifies that this claim is based on a misunderstanding and, in fact, does not yield any counterexample to the argument. Consequently, the classfication established in [2] remains fully valid as stated.

Summary:

- **Proposition. 6.2 in [2]:**
  For an isoparametric hypersurface M in S^{13} with six principal curvatures $(g, m) = (6, 2)$, any *shape operators of a focal submanifold* –consisting of two-parameter family of isospectral operators– satisfy a specific geometric condition, denoted here by [P], as formulated in the proposition.
- **Claim in the Appendix of [3]:**
  The Appendix considers a two-parameter family of isospectral operators $L(t, s)$ and shows that this family violates condition [P]. It is then asserted that $L(t, s)$ provides a counterexample to Proposition 6.2 in [2].
- **Correct conclusion:**
  Although the operators $L(t, s)$ are isospectral, they fail to satisfy the geometric condition [P]. Therefore, they *cannot arise* as shape operators of any focal submanifold. Consequently, the family $L(t, s)$ does not provide a counterexample to Proposition 6.2 in [2].

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The author believes that readers who are well acquainted with the arguments developed in the §§3–6 of [2] readily see that the proof remains sound and does not contradict the computations presented in [3]. For a broader audience, however, the underlying reasoning is too intricate –and at certain moments too delicate– to reproduce in full detail here. Instead, the author highlights several essential points.

2 A brief detail

The central pillar of the author's classification proof is a well-known criterion (see §15 of [1]):
An isoparametric hypersurface M with $(g, m) = (6, 2)$ is homogeneous if and only if the shape operators of a focal submanifold –forming a two-parameter family of isospectral operators– share common kernels.

The key step, therefore, is to show that the shape operators of a focal submanifold must, in fact, have *constant kernels.*

From a purely linear-algebraic perspective, there indeed *exist many two-parameter families of isospectral operators with a wide range of possible kernels*, including the particular family exhibited in [3]. However, when such operators arise as shape operators from geometry, the situation is far more rigid: *only very special kernel configurations are compatible* with the underlying geometric constraints.

In the author's work, a detailed analysis is carried out of how the kernel vector behaves as one moves through the two-parameter family of shape operators intrinsic to the focal submanifold. This variation is controlled not by abstract algebraic freedom but by specific geometric quantities, such as $\nabla_{e_{\underline{6}}} e_{\underline{3}}$, $\nabla_{e_{\underline{3}}} e_{\underline{6}}$ together with their higher derivatives, where $e_{\underline{3}}$ denotes a kernel vector, and $e_{\underline{6}}$ corresponds to parameters governing the family of shape operators. These differential-geometric relations—worked out carefully in §§3–6 of [2]– rigidly determine how the kernel can vary and ultimately force it to remain constant.

By contrast, the Appendix of [3] contains no comparable geometric analysis. Its argument proceeds entirely within the realm of linear algebra and, moreover, the interpretation of the operators constructed there does not correctly reflect the geometric setting of shape operators on a focal submanifold. Consequently, the operators in [3] cannot serve as counterexamples to the results established in [2].

3 Essential points

1. *Isospectrality of the shape operators, taken by itself, is not sufficient.* Although one can construct many two-parameter families of operators sharing the same spectrum, such linear-algebraic families do not necessarily correspond to the geometry of a focal submanifold. The classification argument in [2] relies on geometric constraints that go well beyond the spectral data.
2. *The kernel of each shape operator carries genuine geometric information.*

In the geometric setting, the kernel is not an arbitrary eigenspace but reflects how each kernel vector moves as an operator varies. This behavior encodes the structure of the focal submanifold, and it is precisely this variation that imposes strong restrictions absent from general isospectral families.

3. *The arguments developed in §§3–6 of [2] are indispensable and cannot be reproduced by linear-algebraic considerations alone.*

   These sections analyze the geometric behavior of the shape operators in detail, revealing differential constraints that do not appear in a purely algebraic framework. This geometric analysis is essential for the classification and is exactly what distinguishes the true shape operators of a focal submanifold from artificially constructed isospectral examples.

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References

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